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A NECESSARY AND SUFFICIENT CONDITION FOR FIBERING A MANIFOLD

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§0. INTRODUCTION AND STATEMENT OF RESULTS

THE PURPOSE of this paper is to clarify the following question: Suppose that the *PL* manifold M^n has the homotopy type of the total space of a fiber bundle $\pi: E \to Z$ where the fiber K^k is a topological k-manifold. Can we then fiber M^n itself by K^k, i.e. find some bundle over a manifold V^{n-k} with fiber K^k and total space M^n? (Throughout, we assume that the group of self-homeomorphisms of K^k is the structure group of the fiber bundle.)

We find that, in addition to the aforesaid condition that M^n have the homotopy type of the total space of a K^k bundle (which condition is, trivially, necessary) there are several other conditions which, taken together, suffice to make M^n homeomorphic to the total space of a K^k bundle (provided $n-k$ is odd). These conditions are necessary as it turns out, the necessity being made obvious by a very simple proof.

We note that if M^n is, in fact, homotopy equivalent to the total space of a K^k bundle it follows[8] that the base space Z must be a Poincaré duality space of dimension $n-k$. We therefore impose the additional conditions in order to make sure that the homotopy type of Z is realized by a *PL* manifold V^{n-k} in such a way that $M_0{}^n$, the total space of the induced K^k bundle over V^{n-k} has the same *PL* homeomorphism type as M^n.

We assume throughout that M^n, K^k are 4-connected in order to avoid triangulation difficulties (among other things).

The results of this paper are similar to those announced in [6]. However, we note that in [6], the sufficient conditions for fibering M^n by K^k were a bit more artificial and the necessity of those conditions was unclear, except in special cases.

In order to state our results clearly, we shall need to develop some terminology.

0.1. *Definition.* Let W^r be a compact *PL* r-manifold with boundary. A *shrinking* of W^r is a codimension-0 submanifold $W_0{}^r$ of the interior of W^r such that the closure of $W^r - W_0{}^r$ is a trivial h-cobordism between ∂W^r and $\partial W_0{}^r$. If V^{r-1} is a codimension-0 submanifold of ∂W^r, a *relative shrinking* of W^r, V^{r-1} is a codimension-0 submanifold $W_0{}^r$ such that $W_0{}^r \cap \partial W^r = V_0{}^{r-1}$ is a shrinking of V^{r-1} and so that closure $W^r - W_0{}^r$ is a trivial h-cobordism between closure $(\partial W^r - V^{r-1})$ and closure $(\partial W_0{}^r - V_0{}^{r-1})$.

Now let W^r be a *PL* manifold, U^s a submanifold with *PL* normal bundle. Let L be a triangulated space and let $f: L \to W^r$.

0.2. *Definition.* f is *simplex-wise transverse* to U^s if and only if $f|\sigma$ is *PL* transverse to U^s as a map of *PL* manifolds for every simplex σ of L.

Now let $\bar{U}$ be a fixed *PL* tubular neighborhood of U^s.

0.3. *Definition.* We say that $f: L \to W^r$ is *simplex-wise transverse* to U^s *respecting the tube* $\bar{U}$ if and only if f is simplex wise transverse to U^s and, in addition $\bar{U}_\sigma = f^{-1}(\bar{U}) \cap \sigma$ is a tubular neighborhood of $U_\sigma = f^{-1}(U^s) \cap \sigma$, with $f|\bar{U}_\sigma \to \bar{U}$ a map of *PL* disc bundles, for every simplex σ of L.

Before stating our main result, we notice the following fact: Suppose the closed combinatorial manifold M^n has the homotopy type of the total space W of a topological fiber bundle with fiber K^k over some finite complex. It is easily seen, without loss of generality, that we may replace W by some *PL* q-disc bundle over M^n which we denote W^{n+q}, the superscript denoting its dimension as a manifold. For let $\pi: W \to Z$ be the original K^k bundle. We may replace Z by its stable regular neighborhood in Euclidean space and thereby assume that Z is a compact *PL* manifold-with-boundary of arbitrarily large dimension. W is then seen to be a compact manifold-with-boundary and is combinatorially triangulable since the Kirby–Siebenmann obstraction [4] vanishes. Let $q = \dim W - n$ (which we take as large as we please, compared to n). Then, by any of the well-known embedding theorems, we may assume the homotopy equivalence $M^n \to W^{n+q}$ is a locally-flat embedding and that W^{n+q} is a PL q-disc bundle over M^n.

We now state our main result: Let M^n be a closed 4-connected PL manifold, K^k a closed 4-connected topological manifold, and let $n-k$ be odd and ≥ 5.

THEOREM A. *The following are necessary and sufficient conditions for M^n to be the total space of a K^k-bundle over a closed PL manifold V^{n-k}.*

(i) *There is a PL q-disc bundle over M^n with total space W^{n+q} such that W^{n+q} is also the total space of a K^k bundle over the parallelizable PL manifold-with-boundary Z^{n+q-k}, where $\pi: W^{n+q} \to Z^{n+q-k}$ is the projection map.*

(ii) *Z^{n+q-k} admits a triangulation and a shrinking $Z_0{}^{n+q-k}$ so that*

(a) *For any simplex σ of Z^{n+q-k}, $Z_0{}^{n+q-k} \cap \sigma$ is a codimension-0 submanifold of σ.*

(b) *W^{n+q} may be triangulated so that $\pi^{-1}\sigma$ is a subcomplex for any simplex $\sigma \subseteq Z^{n+q-k}$.*

(c) *$\pi^{-1}Z_0{}^{n+q-k} = W_0{}^{n+q}$ is a tubular neighborhood of M^n in W^{n+q} and, for the triangulation specified above, the identity map $W^{n+q} \to W^{n+q}$ is simplex-wise transverse to M^n respecting the tube $W_0{}^{n+q}$.*

(iii) *There is a deformation $\beta:(W^{n+q}, \partial W^{n+q}) \times I \to W^{n+q}, \partial W^{n+q}$ so that (setting $\beta_t(w) = \beta(w, t)$)*

(a) *β_0 is the identity; $\pi' = \pi \circ \beta_1$ is simplicial.*

(b) *β is simplex-wise transverse to M^n, respecting the tube $W_0{}^{n+q}$ (with respect to some product triangulation on $W^{n+q} \times I$).*

COROLLARY B. *The condition that $n-k$ be odd may be replaced by the assumption that $n-k$ is even and a certain surgery obstruction in $\pi_{n-k}(G/PL)$ vanishes. In any case,* (i), (ii), (iii) *are still necessary irrespective of the vanishing of the surgery obstruction.*

Note that by the remarks preceding the statement of Theorem A, condition (i) is merely equivalent to M^n having the homotopy type of the total space of a K^k bundle.

The idea of the proof of Theorem A (or, at least, the more important part, showing the sufficiency of conditions (i), (ii), (iii)) is to show that Z^{n+q-k} has the homotopy type of an $n-k$ manifold V^{n-k}, by showing that its Spivak normal bundle (recall that Z^{n+q-k} is a $n-k$ dimensional Poincaré space) admits a reduction to a PL bundle. This is accomplished by appeal to the result of the author and J. Morgan[7] that reducing a spherical fibration to a PL bundle is equivalent to putting a "Poincaré transversality structure" on it. Condition (ii) above guarantees the existence of such a structure. Condition (iii) essentially guarantees that the corresponding transversality structure on the Spivak fibration of the total space of the K^k bundle over W over Z (regarding W as an n-dimensional Poincaré space) is essentially that coming from the homotopy equivalence between W and the PL manifold M^n. Thus, by Browder–Novokov–Sullivan theory, Z has the homotopy type of a manifold V^{n-k} and, furthermore the K^k-bundle $M_0{}^n$ over V^{n-k} is homotopy equivalent to M^n by a homotopy equivalence with vanishing normal invariant; thus $M_0{}^n$ is PL homeomorphic to M^n, and M^n is fibered by K^k over V^{n-k}.

§1. NECESSITY OF CONDITIONS (i)-(iii)

As we have noted before, the fact that the fibering of M^n by K^k implies (i), (ii), (iii) is straightforward to verify, involving nothing deeper than standard general position theorems. For the sake of completeness, we include a proof.

Suppose M^n is the total space of a K^k bundle over V^{n-k} with projection $\pi_0: M^n \to V^{n-k}$ where V^{n-k} is a PL manifold. Let Z^{n+q-k} be the stable tubular neighborhood of the embedding of V^{n-k} in $\mathbb{R}^{n+q-k}$, q large; thus Z^{n+q-k} is the total space of a PL q-disc bundle over V^{n-k}. We may take the K^k-bundle over Z^{n+q-k} induced by the projection of Z^{n+q-k} onto V^{n-k} and find that its total space W^{n+q} is, itself, the total space of the PL q-disc bundle over M^n induced by $\pi_0: M^n \to V^{n-k}$. So we have

$$\begin{array}{ccc} M^n & \subseteq & W^{n+q} \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} \\ V^{n-k} & \subseteq & Z^{n+q-k} \end{array}$$

therefore, condition (i) is verified.

We triangulate Z^{n+q-k} and find a shrinking $Z_0^{n+q-k} \subsetneq Z^{n+q-k}$ so that the identity map on Z^{n+q-k} is transverse to V^{n-k} respecting the tube Z_0^{n+q-k}. We also note, by virtue of the Kirby–Siebenmann theorem [4], that we may triangulate W^{n+q} so that $\pi^{-1}\sigma$ is a subcomplex and PL submanifold for each simplex σ of the triangulation of Z^{n+q-k}. On the other hand, it is clear that $\pi^{-1}\sigma$ is in general position with respect to $M^n = \pi^{-1}V^{n-k}$. Moreover, if $W_0^{n+q} = \pi^{-1}Z_0^{n+q-k}$, then W_0^{n+q} is a tubular neighborhood of $M^n\pi^{-1}\sigma$. By repeated use of standard theorems concerning the existence and uniqueness of PL regular neighborhoods we may deform the triangulation of W^{n+q} so that the deformation preserves each subset of the form $\pi^{-1}\sigma$ and so that in the new triangulation, each simplex meets M^n transversally; moreover, we may insure that, with this new triangulation in mind, the identity map $W^{n+q} \to W^{n+q}$ is simplex-wise transverse to M^n respecting the tube W_0^{n+q}. Therefore, condition (ii) is verified.

We may now deform the map π, (keeping ∂W^{n+q} in ∂Z^{n+q-k}) to a map which is a simplicial map from the given triangulation of W^{n+q} to the given triangulation of Z^{n+q-k}. Note that we need not first subdivide the triangulation of W^{n+q} because we have constructed this triangulation so that for each simplex τ of W^{n+q}, there is a simplex σ of Z^{n+q-k} so that $\pi(\tau) \subseteq \sigma$. Because π is a fibering we may cover the deformation of π by a deformation of the identity map, i.e. $\beta^0:(W^{n+q}, \partial W^{n+q}) \times I \to W^{n+q}, \partial W^{n+q}$ so that $\beta_0^{\,0} = \text{identity}$, $\pi \cdot \beta_1^{\,0}$ is simplicial. But note that $\pi \circ \beta_1^{\,0}$ is simplex-wise transverse to V^{n-k}, respecting the tube Z_0^{n+q-k}, because $\beta_1^{\,0}|\tau$ is a linear map to some simplex $\sigma \subseteq Z^{n+q-k}$, whereas the identity map on Z^{n+q-k} is simplex-wise transverse to V^{n-k}, respecting the tube Z_0^{n+q-k}. It then follows that $\beta_1^{\,0}$ is simplex-wise transverse to M^n respecting the tube W_0^{n+q}. Since $\beta_0^{\,0} = \text{identity}$ on W^{n+q} also has this property, we may give $W^{n+q} \times I$ the product triangulation (by picking, e.g. an ordering of the vertices) and then deform β^0 rel $W^{n+q} \times \dot{I}$ so that the result β is simplex-wise transverse to M^n, respecting the tube W_0^{n+q}. Thus condition (iii) is verified and the necessity of (i), (ii), (iii) is proved.

§2. SUFFICIENCY OF CONDITIONS (i), (ii), (iii)

By far the deeper and more interesting part of Theorem A is the sufficiency of conditions (i), (ii), (iii) for the fibering of a manifold. We now proceed to prove this.

We have, by assumption, a q-disc bundle W^{n+q} over M^n which fibers as a K^k bundle over Z^{n+q-k}. Our task, obviously, is to find a $PL(n-k)$ manifold of the homotopy type of Z^{n+q-k} which will serve as V^{n-k}.

Note that, in the first place, Z^{n+q-k} itself is a Poincaré duality space of dimension $n-k$. This is because Z^{n+q-k} is the base of a fibration whose total space and fiber, respectively, have the homotopy type of an n-dimensional Poincaré duality space and a k-dimensional Poincaré duality space. Therefore, by an observation of Quinn [8], Z^{n+q-k} is an $(n-k)$-dimensional Poincaré duality space.

Before we proceed further, we must introduce a certain amount of terminology and, in addition, review some results of [2] and [7], which are also to be found, in somewhat different form, in [3] and [8].

2.1. *Definition.* A homotopy tube of a $(q-1)$-spherical fibration is a pair of spaces R, S, S closed in R, such that the inclusion map $S \subseteq R$ has the $(q-1)$-sphere S^{q-1} as its abstract homotopy theoretic fiber [9]. If there is no need to refer explicitly to the dimension of the fiber, we shall merely call R, S a homotopy tube.

2.2 *Definition.* We shall say that a space A contains R, S as an embedded homotopy tube iff R, S is a homotopy tube and $A = R \cup Q$ where R, Q are closed and $S = R \cap Q$.

Example. Let $S: E \to B$ be an S^{q-1}-fibration; then the pair $\mathcal{M}_\xi$, E, (where $\mathcal{M}_\xi$ denotes mapping cylinder) is a homotopy tube. The Thom space $T(\xi) = \mathcal{M}_\xi \underset{E}{\cup} cE$ contains $\mathcal{M}_\xi$, E as an embedded homotopy tube. (Here cE denotes unreduced cone on E).

Example. Let P^{n+q} be a PL thickening of an n-dimensional Poincaré duality space, $q \geq 3$. Then P^{n+q}, ∂P^{n+q} is a homotopy tube of a $(q-1)$-spherical fibration. If P_0^{n+q} is a shrinking of P^{n+q}, then P^{n-q} contains P_0^{n+q}, ∂P_0^{n+q} as an embedded homotopy tube.

Suppose now that A contains R, S as an embedded homotopy tube (of a $(q-1)$-spherical fibration). Let X^n be a PL n-manifold (possibly with boundary).

2.3. *Definition.* We shall say that $f: X^n \to A$ is *Poincaré transverse* (abbreviated "*PT*") to the embedded homotopy tube R, S iff: $f^{-1}(R) = \emptyset$ or $f^{-1}(R) = D$ is a codimension 0 submanifold of X^n

with $D_1 = D \cap \partial X^n$ a codimension-0 submanifold of ∂X^n and of ∂D; $f^{-1}(S) = B$ = closure $\partial D - D_1$ is a codimension-0 submanifold of ∂D with $B_1 = \partial B = \partial D_1 = B \cap \partial X^n$: $(D, B), (D_1, B_1)$ are homotopy tubes, making the diagram

$$\begin{array}{ccccc} B_1 & \xrightarrow{\subseteq} & B & \xrightarrow{f} & S \\ \Big|\cap & & \Big|\cap & & \Big|\cap \\ D_1 & \xrightarrow{\subseteq} & D & \xrightarrow{f} & R \end{array}$$

into a sequence of maps of $(q-1)$-spherical fibrations (i.e., $B \xrightarrow{\subseteq} D$ is, up to homotopy, the $(q-1)$-spherical fibration induced from $S \xrightarrow{\subseteq} R$ by $f|D \to R$, (and likewise if we replace D, B by D_1, B_1)).

2.4. *Definition.* Let A contain R, S as an embedded homotopy tube; let L be a triangulated space, and let $f: L \to A$. We say that f is *simplex-wise PT to the embedded homotopy tube R, S* iff $f|\sigma$ is PT to the embedded homotopy tube R, S for each simplex σ of L.

Now let $\xi: E \to C$ be a $(q-1)$-spherical fibration over a complex. Let $s: \Delta^r \to T(\xi)$ be a map from the standard r-simplex Δ^r. We shall say that s is a PT singular simplex if s is PT to the embedded homotopy tube $\mathcal{M}_\xi$, E, as is the restriction of S to all lower-dimensional faces of Δ^r. The set of PT singular simplices forms a subcomplex of the singular complex $S(T(\xi))$. (See [5, 7]). Letting $\omega(\xi)$ also denote the geometric realization of this complex, we have a map

$$p: \omega(\xi) \to T(\xi).$$

Now assume that the base space C of ξ is a 4-connected, finite dimensional complex and that q is large compared to dim C. We ask whether the map $p: \omega(\xi) \to T(\xi)$ admits a section, up to homotopy, and what homotopy classes of such sections correspond to. We first observe ([2, 6, 7]) that a homotopy section of p (or, what amounts to the same thing, a deformation of $S(T(\xi))$ into $\omega(\xi)$) yields a "Poincaré transversality structure" on ξ, that is, a consistent method for deforming any map $f: L \to T(\xi)$ of a triangulated space L to $T(\xi)$ to a map $f': L \to T(\xi)$ which is simplex-wise PT to the embedded homotopy tube $\mathcal{M}_\xi$, E. The important fact is that such transversality structures correspond to PL structures on ξ. We state this as a proposition.

2.5. PROPOSITION. *Homotopy classes of homotopy sections of the map* $p: \omega(\xi) \to T(\xi)$ *are in* 1·1 *correspondence with homotopy classes of liftings k in the diagram*

$$\begin{tikzcd} & BPL(q) \arrow[d] \\ C \arrow[ur, dashed, "k"] \arrow[r, "\xi"] & BG(q) \end{tikzcd}$$

(N.B.: We think of a homotopy section of p as a map $t: T(\xi) \to \omega(\xi)$, together with a deformation of the identity on $T(\xi)$ to $p \circ t$, two such being homotopic not merely if the underlying maps into $\omega(\xi)$ are homotopic; the deformations must also be homotopic rel the identity map on $T(\xi)$.)

Half of Proposition 2.5 is easily seen to be true. If ξ does admit the structure of a PL bundle, then applying the PL version of the Thom transversality theorem gives us a consistent method of deforming any map $f: L \to T(\xi^k)$ to a simplex-wise PL transverse map $f': L \to T(\xi^k)$, which is, automatically, simplex-wise PT. The important point, then, of Proposition 2.5 is that, up to homotopy, any section of p arises in this way.

Now suppose $\bar{P}^{n+q}$ is a PL thickening of an n-dimensional Poincaré duality space, where P^n is 4-connected $n \geq 5$, and q is large compared to n. Let $\bar{P}_0^{n+q}$ be a shrinking of $\bar{P}^{n+q}$, making $\bar{P}_0^{n+q}$, $\partial\bar{P}_0^{n+q}$ into an embedded homotopy tube. Fix some triangulation of $\bar{P}^{n+q}$.

Consider deformations α_t, $t \in [0, 1]$ of the identity map $= \alpha_0$ on $\bar{P}^{n+q}$, $\partial\bar{P}^{n+q}$ to some map α_1: $\bar{P}^{n+q}, \partial\bar{P}^{n+q} \to \bar{P}^{n+q}, \partial\bar{P}^{n+q}$.

2.6. *Definition.* Such a deformation will be called a *nice* deformation (abbreviated n-deformation) if α_1 is simplex-wise PT to the embedded homotopy tube $\bar{P}_0^{n+q}$, $\partial\bar{P}_0^{n+q}$. Call two such n-deformations α_t^0, α_t^1 n-equivalent iff there is a map

$$\beta: (\bar{P}^{n+q}, \partial\bar{P}^{n+q}) \times I \times I \to \bar{P}^{n+q}, \partial\bar{P}^{n+q}$$

such that:
$$\beta(x,t,0)=\alpha_t^{\,0}(x)$$
$$\beta(x,t,1)=\alpha_t^{\,1}(x)$$
$$\beta(x,0,s)=x=\alpha_0^{\,0}(x)=\alpha_0^{\,1}(x)$$

and $\beta\,|\,\bar P^{n+q}\times\{1\}\times I$ is simplex-wise PT to $\bar P_0^{\,n+q}$, $\partial\bar P_0^{\,n+q}$ (for some triangulation of $\bar P^{n+q}\times\{1\}\times I$ extending the obvious one on $\bar P^{n+q}\times\{1\}\times \dot I$).

Now let ξ denote the $(q-1)$-spherical fibration over P^n corresponding to the inclusion $\partial\bar P^{n+q}\xrightarrow{\subseteq}\bar P^{n+q}$ (or $\partial\bar P_0^{\,n+q}\xrightarrow{\subseteq}\bar P_0^{\,n+q}$). We claim that an n-deformation α_t of $\bar P^{n+q}$, $\partial\bar P^{n+q}$ determines a section of $p:\omega(\xi)\to T(\xi)$ and that n-equivalent deformations determine homotopic sections.

Proof. We may construct a standard homotopy equivalence between $T(\xi)$ and the space $T=\bar P_0^{\,n+q}\cup c\partial\bar P_0^{\,n+q}$. Moreover, we may also think of T as $\bar P^{n+q}\cup c\partial\bar P^{n+q}$ where $\bar P_0^{\,n+q}\subseteq \operatorname{int}\bar P^{n+q}$ and $c\partial\bar P^{n+q}\subseteq \operatorname{int} c\partial\bar P_0^{\,n+q}$. T contains $\bar P_0^{\,n+q}$, $\partial\bar P_0^{\,n+q}$ as an embedded homotopy tube. The deformation $\dot\alpha_t=\alpha_t|\partial\bar P^{n+q}\to\partial\bar P^{n+q}$ extends to $c\dot\alpha_t:c\partial\bar P^{n+q}\to c\partial\bar P^{n+q}$ in the obvious way so that α_t extends to $\hat\alpha_t:T\to T$, where $\hat\alpha_t=\alpha_t\cup c\alpha_t$. Now if T is triangulated in the obvious way, extending the given triangulation of $\bar P^{n+q}$, it becomes clear that $\hat\alpha_1$ is simplex-wise PT to the embedded homotopy tube $\bar P_0^{\,n+q}$, $\partial\bar P_0^{\,n+q}$. Thus the map $T\xrightarrow{\hat\alpha_1}T\xrightarrow{\simeq}T(\xi)$ is seen to lift to $\omega(\xi)$. One may relativize the above argument to check that if n-deformations are n-equivalent, the sections of p determined in this way are n-equivalent.

Example. Let P^n, $\bar P^{n+q}$, $\bar P_0^{\,n+q}$ be as above, and suppose that $M_1^{\,n}$, $M_2^{\,n}$ are PL manifolds and $s_i:M_i^{\,n}\to P^n$, $i=1,2$, are homotopy equivalences. We may regard $M_1^{\,n}$ (resp., $M_2^{\,n}$) as embedded in the interior of $\bar P_0^{\,n+q}\subseteq\bar P^{n+q}$, in which case, $\bar P_0^{\,n+q}$ is a PL tubular neighborhood. Now it is clear that we may deform the identity map $\bar P^{n+q}$, $\partial\bar P^{n+q}-\bar P^{n+q}$, $\partial\bar P^{n+q}$ to a map which is simplex-wise transverse to $M_1^{\,n}$ (resp. $M_2^{\,n}$) respecting the tube $P_0^{\,n+q}$. Obviously, a map which is simplex-wise transverse to $M_1^{\,n}$ respecting the tube $\bar P_0^{\,n+q}$ satisfies the weaker condition of being simplex-wise PT to the embedded homotopy tube $\bar P_0^{\,n+q}$, $\partial\bar P_0^{\,n+q}$. Thus the deformation in question is an n-deformation. An easy application of PL transversality theorems suffices to show that any two n-deformations arising thus are n-equivalent. So let $\alpha_t^{\,1}$ be anything in this n-equivalence class and let $\alpha_t^{\,2}$ be any n-deformation obtained by substituting $M_2^{\,n}$ for $M_1^{\,n}$ in the procedure above.

We may then ask whether $\alpha_t^{\,1}$ is n-equivalent to $\alpha_t^{\,2}$. A positive answer to this question implies that the two "transversality structures" on ξ are the same, i.e. $\alpha_t^{\,1}$ and $\alpha_t^{\,2}$ yield the same reductions of ξ to a PL bundle. Thus, the n-equivalence of $\alpha_t^{\,1}$ and $\alpha_t^{\,2}$ implies the vanishing of the normal invariant in $[P^n, G/PL]$ which is a function of the difference between the two homotopy triangulations $M_1^{\,n}$, s_1 and $M_2^{\,n}$, s_2 of P^n (cf. [10]). Therefore, if $\alpha_t^{\,1}$ is n-equivalent to $\alpha_t^{\,2}$, $M_1^{\,n}$ is PL homeomorphic to $M_2^{\,n}$. (We also claim, without proof, that the converse holds, i.e. the vanishing of the normal invariant will imply the n-equivalence of $\alpha_t^{\,1}$ and $\alpha_t^{\,2}$).

We are now finally ready to proceed with the proof, *per se*, of the sufficiency of conditions (i), (ii), (iii):

We have, as part of the hypotheses, fixed a triangulation of Z^{n+q-k}. Clearly, Z^{n+q-k} contains $Z_0^{\,n+q-k}$, $\partial Z_0^{\,n+q-k}$ as an embedded homotopy tube. We first wish to show that the identity map $Z^{n+q-k}\to Z^{n+q-k}$ is simplex-wise PT to the embedded homotopy tube $Z_0^{\,n+q-k}$, $\partial Z_0^{\,n+q-k}$. To do this, it will be sufficient to check that, for any r-simplex σ_r of Z^{n+q-k}, either $\sigma^r\cap Z_0^{\,n+q-k}=\emptyset$ or $\sigma^r\cap\partial Z_0^{\,n+q-k}\subseteq\sigma^r\cap Z_0^{\,n+q-k}$ is, up to homotopy, a $(q-1)$-spherical fibration induced from $\partial Z_0^{\,n+q-k}\subseteq Z_0^{\,n+q-k}$. But notice that if $\sigma^r\cap Z_0^{\,n+q-k}\neq\emptyset$, then $A_{\sigma^r}=\pi^{-1}(\sigma^r\cap Z_0^{\,n+q-k})$ is a PL tubular neighborhood of $M^n\cap\pi^{-1}\sigma^r$ in $\pi^{-1}\sigma^r$, having boundary $\pi^{-1}(\sigma^r\cap\partial Z_0^{\,n+q-k})=B_{\sigma^r}$.

We may thus write down a diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & K^k \arrow[r, "="] \arrow[d] & K^k \arrow[d] \\
S^{q-1} \arrow[r] \arrow[d] & B_{\sigma^r} \arrow[r, "\subseteq"] \arrow[d, "\pi"] & A_{\sigma^r} \arrow[d, "\pi"] \\
F \arrow[r] & \sigma^r\cap\partial Z_0^{\,n+q-k} \arrow[r, "\subseteq"] & \sigma^r\cap Z_0^{\,n+q-k}
\end{tikzcd}$$

in which each short vertical or horizontal sequence is, up to homotopy, fiber into total space into base. We see that $B_{\sigma^r} \subseteq A_{\sigma^r}$ includes a PL $(q-1)$-sphere bundle into its associated disc bundle, and the fiber is, of course, S^{q-1}. $\pi|B_{\sigma^r}$ and $\pi|A_{\sigma^r}$ are, of course, fiberings with fiber K^k by definition of A_{σ^r} and B_{σ^r}. The fiber $*$ of $K^k \xrightarrow{=} K^k$ is, of course, contractible, but then, since $*$ is also the fiber of $S^{q-1} \to F$, it follows that F has the homotopy type of S^{q-1}. It is trivial to check further that $\sigma^r \cap \partial Z_0^{n+q-k} \subseteq \sigma^r \cap Z_0^{n+q-k}$ is induced, as a $(q-1)$-spherical fibration, from the fibration corresponding to $\partial Z_0^{n+q-k} \subseteq Z_0^{n+q-k}$. Thus our assertion that the identity map on Z^{n+q-k} is simplex-wise PT to Z_0^{n+q-k}, ∂Z_0^{n+q-k}, is established.

Thus the trivial deformation of the identity on Z^{n+q-k} is an n-deformation. If we denote the S^{q-1}-fibration corresponding to $\partial Z_0^{n+q-k} \subseteq Z_0^{n+q-k}$ by ξ, this n-deformation yields a reduction of ξ to a PL bundle, which we may interpret as a homotopy equivalence

$$e_0: Z_0^{n+q-k}, \partial Z_0^{n+q-k} \to D(\xi_0), S(\xi_0)$$

where ξ_0 is some q-dimensional PL bundle over a base space B_Z having the homotopy type of Z_0^{n+q-k}. Let us now assume that e_0 is PL transverse to B_Z, with $e_0^{-1}B_Z = V_0^{n-k}$ having PL normal bundle ν in Z_0^{n+q-k}. Note that $V_0^{n-k} \to B_Z$ is a degree 1 map covered by a PL bundle map $\nu \to \xi_0$. Since Z_0^{n+q-k} is parallelizable, ν is, stably, the normal bundle of V_0^{n-k} in Euclidean $(n+q-k)$-space.

Thus, if the surgery invariant of this normal map, which lives in $\pi_{n-k}(G/PL)$, is trivial, then, assuming q is large enough, we may deform e_0 to a new map e so that $e^{-1}B_Z = V^{n-k}$ is homotopy equivalent to B_Z, with $e|V^{n-k}$ the homotopy equivalence (see [1]).

But we observe that, without loss of generality, q may be assumed to be as large as we please. (If not, replace Z^{n+q-k}, W^{n+q} by their respective products with D^r for large enough r and verify that hypotheses (i)–(iii) may still be made to hold.) Also, we assume that $n-k$ is odd so:

(S) The surgery obstruction automatically vanishes.

(We single out this observation for future reference in the subsequent proof of Corollary B.)

Thus V^{n-k} lies in Z_0^{n+q-k} as a submanifold and a deformation retract; hence Z_0^{n+q-k} is a tubular neighborhood of V^{n-k}.

We digress to extend our terminology a bit. Again, consider $\bar{P}^{n+q}$, a codimension q thickening of a Poincaré duality space P^n, and let $\bar{P}_0^{n+q}$ be a shrinking; set $T = \bar{P}_0^{n+q} \cup c\partial\bar{P}_0^{n+q} = \bar{P}^{n+q} \cup c\partial\bar{P}^{n+q}$. Let C be a triangulated space, and $f: C \to T$. An n-deformation of f is a homotopy F_t, with $F_0 = f$ such that F_1 is simplex-wise PT to the embedded homotopy tube $\bar{P}_0^{n+q}$, $\partial\bar{P}_0^{n+q}$. Call two such n-deformations F_t^0, F_t^1 n-equivalent if there is a map $E: C \times I \times I \to T$ so that

$$E(x, t, 0) = F_t^0(x)$$

$$E(x, t, 1) = F_t^1(x)$$

$$E(x, 0, s) = f(x) = F_0^0(x) = F_0^1(x)$$

and $E|C \times \{1\} \times I$ is simplex-wise PT to the homotopy tube $\bar{P}_0^{n+q-k}$, $\partial\bar{P}_0^{n+q-k}$. Note that if f itself is simplex-wise PT to $\bar{P}_0^{n+q}$, $\partial\bar{P}_0^{n+q}$, as is F_t, i.e. the map $F: C \times I \to T$, then F_t is n-equivalent to the trivial deformation of f.

We return to the case at hand. We have noted V^{n-k} has Z_0^{n+q-k} as a tubular neighborhood. We may deform the identity map on $T = Z_0^{n+q-k} \cup c\partial Z_0^{n+q-k} = Z^{n+q-k} \cup c\partial Z^{n+q-k}$ by a deformation H_t such that H_1 is simplex-wise transverse to V^{n-k}, respecting the tube Z_0^{n+q-k}. The important point is this: recall that the identity map on Z^{n+q-k} is simplex-wise PT to the homotopy tube Z_0^{n+q-k}, ∂Z_0^{n+q-k}; thus, so is the identity on T. We claim that the entire deformation $H: T \times I \to T$ may be taken to be simplex-wise PT to Z_0^{n+q-k}, ∂Z_0^{n+q-k} (for the triangulation of $T \times I$ coming from an arbitrary ordering of the vertices of T). In other words, the n-deformation H_t is n-equivalent to the trivial deformation.

Note that $M_0^n = \pi^{-1}V^{n-k}$ is a topological manifold with a unique PL structure: We now must show that M_0^n is PL homeomorphic to M^n. There is an obvious homotopy equivalence $M_0^n \to M^n$ and we shall show that the normal invariant of this homotopy equivalence vanishes in $[M^n, G/PL]$, thereby allowing us to conclude that M_0^n is PL homeomorphic to M^n.

We adopt some notation: As before $T = Z_0^{n+q-k} \cup c\partial Z_0^{n+q-k} = Z^{n+q-k} \cup c\partial Z^{n+q-k}$; set $U = W_0^{n+q} \cup c\partial W_0^{n+q} = W^{n+q} \cup c\partial W^{n+q}$. If α_t is a deformation of the identity map on W, ∂W, we let $\hat{\alpha}_t$ denote the deformation of the identity on U obtained by "coning off", i.e. $\hat{\alpha}_t(x) = \alpha_t(x)$ for $x \in W^{n+q}$, $\hat{\alpha}_t(x, s) = (\alpha_t(x), s)$ for $x \in \partial W^{n+q}$, $s \in [0, 1)$ (i.e. $\partial W^{n+q} \times [0, 1)$ is regarded as a subspace of $c\partial W^{n+q}$), $\hat{\alpha}_t * = *$ for the cone point $*$. We set $\hat{\alpha}: U \times I \to U$ by $\hat{\alpha}(x, t) = \hat{\alpha}_t(x)$. Similarly, if $f: W^{n+q}, \partial W^{n+q} \to Z^{n+q-k}, \partial Z^{n+q-k}$, we set $\hat{f}$ to be the map $U \to T$ obtain by extending in a like manner.

Now consider $\hat{\pi}: U \to T$. We are going to define two n-deformations of $\hat{\pi}$. We obtain the first as follows: let α_t be a deformation of the identity $= \alpha_0$ on W^{n+q}, ∂W^{n+q} so that α_1 is simplex-wise transverse to M_0^n, respecting the tube W_0^{n+q}. Then $\hat{\pi} \cdot \hat{\alpha}_t = A_t$ is the desired n-deformation of $\hat{\pi}$ because A_1 is simplex-wise transverse to V^{n-k} respecting the tube Z_0^{n+q-k}.

We get the second n-deformation as follows: By condition (iii), there is a deformation β_t of the identity map on W^{n+q}, ∂W^{n+q} so that β_t covers a deformation of π to a simplicial map and so that $\beta: W \times I \to W$ is simplex-wise transverse to M^n respecting the tube W_0^{n+q}. It then follows that $\hat{\pi} \cdot \hat{\beta}$ is simplex-wise PT to the embedded homotopy tube Z_0^{n+q-k}, ∂Z_0^{n+q-k}. Furthermore, as we have observed above, there is a deformation H_t of the identity map on T so that H is simplex-wise PT to the homotopy tube Z_0^{n+q-k}, ∂Z_0^{n+q-k} and so that H_1 is simplex-wise transverse to V^{n-k}, respecting the tube Z_0^{n+q-k}. Let $\beta'_t = H_t \circ \hat{\pi} \circ \hat{\beta}_1$ be a deformation of $\hat{\pi} \circ \hat{\beta}_1$. Then, concatenating β'_t with $\hat{\pi} \circ \hat{\beta}_t$ yields a deformation C_t of $\hat{\pi}$. We claim that this is an n-deformation, and, in fact n-equivalent to the trivial, i.e. constant, deformation of $\hat{\pi}$. This is because the first "half" of C_t, i.e. $\hat{\pi} \cdot \hat{\beta}$ was simplex-wise PT to Z_0^{n+q-k}, ∂Z_0^{n+q-k}, while the second half, i.e. β'_t, may be written as $U \times I \xrightarrow{\beta_1 \times \mathrm{id}} T \times I \xrightarrow{H} T$. Now $\beta_1 \times \mathrm{id}$ is simplicial (for suitable triangulations of $U \times I$, $T \times I$), H is simplex-wise PT to Z_0^{n+q-k}, ∂Z_0^{n+q-k}. Therefore the composition is simplex-wise PT to Z_0^{n+q-k}, ∂Z_0^{n+q-k}. (Here, we make use of the fact that if $\Delta^r \to \Delta^s$ is linear and $\Delta^s \to T$ is PT to Z_0^{n+q-k}, ∂Z_0^{n+q-k}, so is the composition $\Delta^r \to \Delta^s \to T$). Finally, we make the additional observation that C_1, the last stage of C_t is, in fact, simplex-wise transverse to V^{n-k} respecting the tube Z_0^{n+q-k}. Therefore C_t is an n-deformation of $\hat{\pi}$ n-equivalent to the trivial n-deformation, as required.

Now we make use of the fact that both A_1 and C_1 are simplex-wise transverse to V^{n-k}, respecting the tube Z_0^{n+q-k}. It is then easy to show, using the PL version of the Thom transversality theorem, that A_t and C_t are n-equivalent n-deformations of $\hat{\pi}$. Therefore A_t is n-equivalent to the trivial n-deformation of $\hat{\pi}$, which we now denote by J_t.

Now observe that if F_t is any n-deformation of the identity of U, $\hat{\pi} \circ F_t$ is an n-deformation of $\hat{\pi}$.

2.7. Proposition. *If G_t, F_t are n-deformations of the identity on U, and $\hat{\pi} \circ G_t$ is n-equivalent to $\hat{\pi} \cdot F_t$, then G_t is n-equivalent to F_t.*

Proof of 2.7. The proof is by appeal to [7] (which is also subsumed in [2]). Let ξ, η denote the $(q-1)$-spherical fibrations corresponding to $\partial Z_0^{n+q-k} \subsetneq Z_0^{n+q-k}$, $\partial W_0^{n+q-k} \subsetneq W_0^{n+q-k}$ respectively. We have a diagram

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & F \arrow[d] \\ \omega(\eta) \arrow[r, "\omega(\pi)"] \arrow[d, "p_1"'] & \omega(\xi) \arrow[d, "p"] \\ T(\eta) \arrow[r, "T(\pi)"] & T(\xi) \end{tikzcd}$$

where F_1, F are, respectively, the fibers of $\omega(\eta) \xrightarrow{p_1} T(\eta)$, $\omega(\xi) \xrightarrow{p} T(\xi)$. The map $F_1 \to F$ induces isomorphism of homotopy groups below dimension q and between dimension $q+4$ and $2q-2$. Given a map $h: C \to T$, we may think of T as $T(\xi)$ and interpret a nice deformation H_t of h as a lifting, which we denote $|H_t|$, in the diagram

$$\begin{tikzcd} & \omega(\xi) \arrow[d] \\ C \arrow[r, "h"] \arrow[ur, dashed, "|H_t|"] & T(\xi). \end{tikzcd}$$

Nicely equivalent deformations correspond to homotopic sections. A similar observation applies if $h: C \to U$.

We have a diagram

$$\begin{tikzcd} & \omega(\eta) \arrow[r, "\omega(\pi)"] \arrow[d, "p_1"] & \omega(\xi) \arrow[d, "p"] \\ U \arrow[ur, bend left, "|F_t|"] \arrow[ur, "|H_t|"'] \arrow[r, "\simeq"] & T(\eta) \arrow[r, "T(\pi)"] & T(\xi) \end{tikzcd}$$

where $\omega(\pi)\circ|F_t| = |\hat{\pi}\circ F_t|$, $\omega(\pi)\circ|G_t| = |\hat{\pi}\circ G_t|$. F_t is n-equivalent to G_t if $|F_t|$ and $|G_t|$ are homotopic sections. But since $\hat{\pi}\circ F_t$ is n-equivalent to $\hat{\pi}\circ G_t$, $|\hat{\pi}\circ G_t|$ is homotopic, as a section, to $|\hat{\pi}\circ F_t|$, and therefore, by standard obstruction theory, $|G_t|$ and $|F_t|$ are homotopic sections. Therefore F_t and G_t are n-equivalent and 2.7 is proved. (N.B.: We need the 4-connectivity of $W^{n+q} \simeq M^n$ in order to get around the difficulty that it is not clear that $F_1 \to F$ is a weak homotopy equivalence between dimensions $q+1$ and $q+3$.)

Returning to the case at hand, we have the trivial n-deformation J_t of $\hat{\pi}$ n-equivalent to A_t. Therefore, the trivial n-deformation I_t of the identity on U is n-equivalent to $\hat{\alpha}_t$.

But note the trivial n-deformation I_t "deforms" the identity on U to a map simplex-wise transverse to M^n while $\hat{\alpha}_t$ deforms the identity to a map simplex-wise transverse to $M_0{}^n$. Furthermore, the difference between the two liftings $|\alpha_t|$ and $|I_t|$ completely characterizes the difference between two PL reductions of the $(q-1)$-spherical fibration η. The difference between these two reductions is an element of $[M^n, G/PL]$ and may be identified with the normal invariant of the homotopy equivalence $M_0{}^n \to M^n$. But since $J_t = \hat{\pi}\circ I_t$ is n-equivalent to $A_t = \hat{\pi}\circ\hat{\alpha}_t$, $\hat{\alpha}_t$ is n-equivalent to I_t and thus, the normal invariant vanishes, thus $M_0{}^n$ is PL homeomorphic to M^n. This completes the proof of Theorem A.

Proof of Corollary B. The proof of the necessity of (i), (ii), (iii) does not depend on $n-k$ being odd.

As for the sufficiency, the proof of Theorem A goes through with $n-k$ even, save for the step marked (S) in the text above, where it was asserted that a surgery obstruction vanished automatically since it lay in an odd-dimensional homotopy group of G/PL. We replace that assertion by the *ad hoc* hypothesis that this surgery assumption in $\pi_{n-k}(G/PL)$ is assumed to be zero.

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